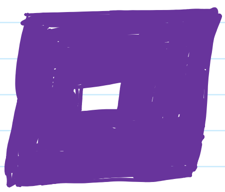
Convolution:-

$$G(i,j) = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H(u,v)\, I(i-u, j-v)$$

$$G = H * I$$

Equivant to flip the filter in both the directions (bottom to top, right to left) and apply cross correlation.

$I(i,j)$

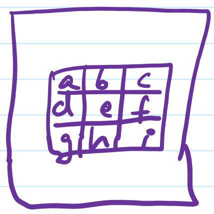


Impulse signal

$* \; H(u,v)$ $\circledast \; H(u,v)$

a b c	i h g
d e f	f e d
g h i	c b a

$G(i,j)$



Linear Shift Invariant Operators:-

Both correlation and convolution are LSIO.

① Linearity (or superposition principle)

$$I \circ (h_0 + h_1) = I \circ h_0 + I \circ h_1$$

② Shift-invariance

Shifting/translating a signal commutes with applying the operator.

lated

with applying the operator.

$$g(i,j) = h(i+k, j+l)$$ Translated g

$\Longleftrightarrow$

a new filter

$$(f \circ g)(i,j) = (f \circ h)(i+k, j+l)$$

effect of operator is same everywhere as long as the image has same characteristics.

Cat at top / bottom.
same output

Convolution properties :-

① Commutation
$$a * b = b * a$$

Conceptually no difference between filter and signal.

② Associativity

$$a * (b * c) = (a * b) * c$$

→ we often apply filters one after the other $(((a * b1) * b2) * b3)$

→ $\Rightarrow$ $a * (b1 * b2 * b3)$

③ Distributive over addition :-

$$a * (b + c) = (a * b) + (a * c)$$

→ We can combine the response of a signal over two or more filters by combining the filters.

④ Scalar factor out

$$k\, a * b = a * kb = k(a * b)$$

⑤ Identity

Unit impulse $e = [\ldots, 0, 0, 1, 0, 0, \ldots]$

$$a * e = a$$

⑥ Separability

Convolution operator requires k^2 operations per pixel, where k is the width (and height) of a convolution kernel.

How to reduce?

Let $k = 3 \Rightarrow 3 \times 3$ 9 operations

$N \times N$ image $\Rightarrow$ cost $N^2 \times 9$

⇒ 1D horizontal convolution followed by a 1D vertical convolution requires 2K operations. ⇒ separability

$N^2 \times 6$

$k = v h^T$

v, h, are 1D kernels and k is the 2D kernel

Example :-

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad v = h = \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\frac{1}{8}\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad v = \frac{1}{4}\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \quad h = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

How to tell if a given kernel k is separable?

look at SVD, and if only one singular value is non-zero, then it is separable

$k = U \Sigma V^T$