8. Projective Transformations (6/2/24)

05 February 2024 12:34

Affine Transformation :-

Rotation & Scaling

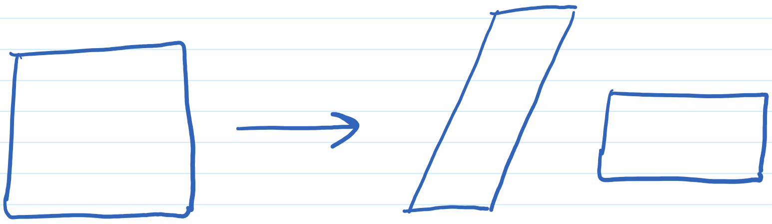
$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

translation

→ Origin changes.

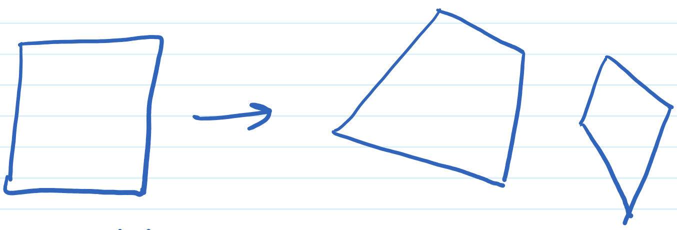
→ Lines map to lines.

→ Parallel lines remain parallel.



Square

Projective Transformation :-



square

→ origin does not necessarily map to origin.

→ Lines maps to lines

→ Parallel lines does not necessarily remain parallel

Any transformation of the form

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

also known as

"Homography"

Number of unknowns $= 8$

$\Rightarrow$ 8 dof.

Computing Homography :-

The homography is a transformation matrix that takes you from one plane to another plane.

Source Image　　　　Destination Image

$$\begin{bmatrix} x_d \\ y_d \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ 1 \end{bmatrix}$$

8 dof $\Rightarrow$ Minimum no. of matching points we need $= 4$

For a given pair i of corresponding points :-

$$\left.\begin{aligned} x_d^{(i)} &= \frac{h_{11} x_s^{(i)} + h_{12} y_s^{(i)} + h_{13}}{h_{31} x_s^{(i)} + h_{32} y_s^{(i)} + h_{33}} \\ y_d^{(i)} &= \frac{h_{21} x_s^{(i)} + h_{22} y_s^{(i)} + h_{23}}{h_{31} x_s^{(i)} + h_{32} y_s^{(i)} + h_{33}} \end{aligned}\right\} \text{— (1)}$$

$$\left.\begin{aligned} x_d^{(i)} &= \frac{h_{11} x_s^{(i)} + h_{12} y_s^{(i)} + h_{13}}{h_{31} x_s^{(i)} + h_{32} y_s^{(i)} + h_{33}} \\ y_d^{(i)} &= \frac{h_{21} x_s^{(i)} + h_{22} y_s^{(i)} + h_{23}}{h_{31} x_s^{(i)} + h_{32} y_s^{(i)} + h_{33}} \end{aligned}\right\} \text{— (1)}$$

Rearranging (1),

$$\left.\begin{aligned} x_d^{(i)}\left(h_{31} x_s^{(i)} + h_{32} y_s^{(i)} + h_{33}\right) &= h_{11} x_s^{(i)} + h_{12} y_s^{(i)} + h_{13} \\ y_d^{(i)}\left(h_{31} x_s^{(i)} + h_{32} y_s^{(i)} + h_{33}\right) &= h_{21} x_s^{(i)} + h_{22} y_s^{(i)} + h_{23} \end{aligned}\right\}$$

— (2)

Writing as matrices,

$$\underbrace{\begin{bmatrix} x_s^{(i)} & y_s^{(i)} & 1 & 0 & 0 & 0 & -x_d^{(i)} x_s^{(i)} & -x_d^{(i)} y_s^{(i)} & -x_d^{(i)} \\ 0 & 0 & 0 & x_s^{(i)} & y_s^{(i)} & 1 & -y_d^{(i)} x_s^{(i)} & -y_d^{(i)} y_s^{(i)} & -y_d^{(i)} \end{bmatrix}}_{\text{knowns}} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{\text{unknowns}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{— (3)}$$

Minimum 8 equations are needed to solve

$\Rightarrow$ 4 pairs of matching points.

Source Image — Destination Image

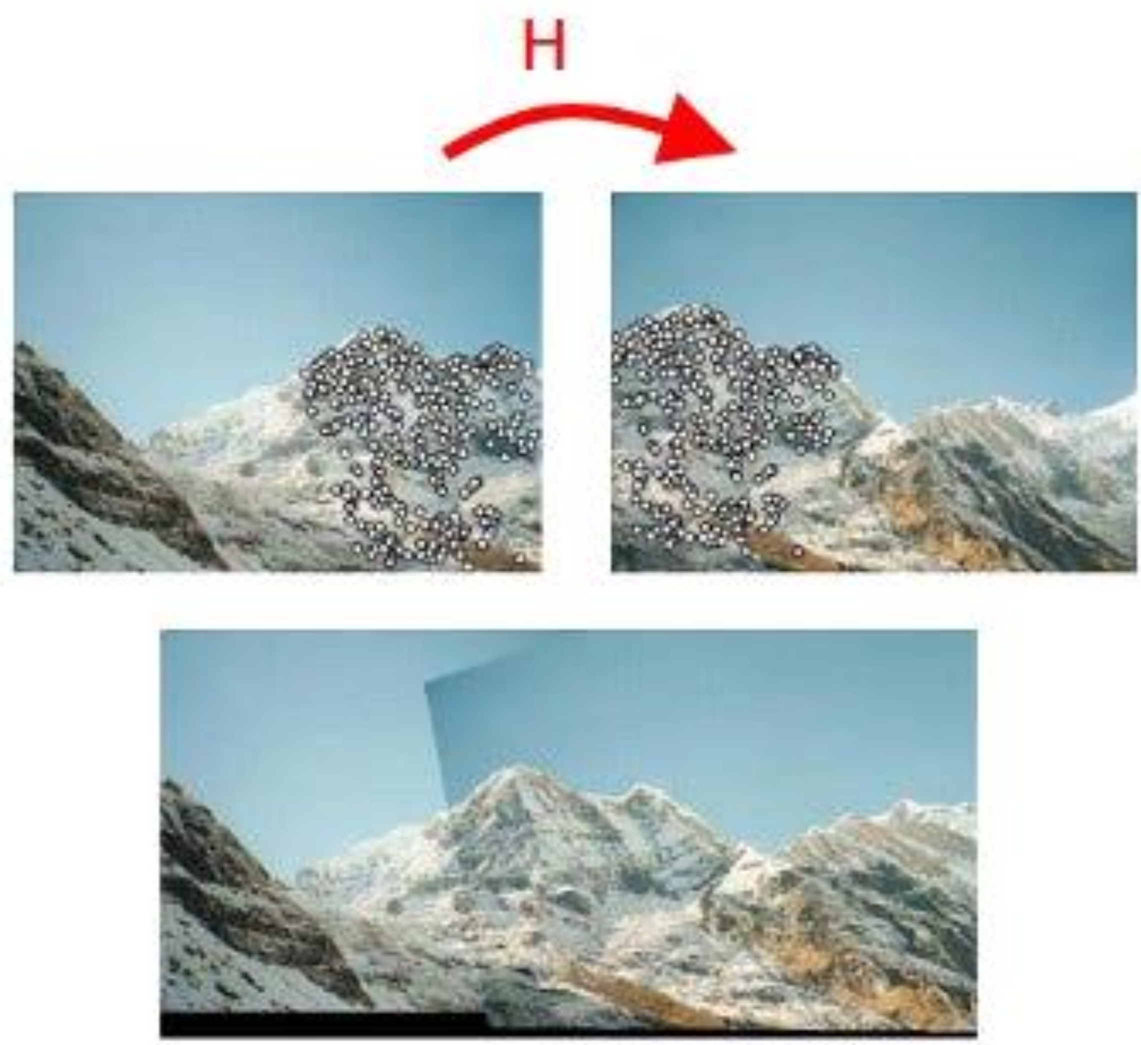
Let 4 pairs are,

$$(x_1^1, y_1^1) \longrightarrow (x_2^1, y_2^1)$$
$$(x_1^2, y_1^2) \longrightarrow (x_2^2, y_2^2)$$
$$(x_1^3, y_1^3) \longrightarrow (x_2^3, y_2^3)$$
$$(x_1^4, y_1^4) \longrightarrow (x_2^4, y_2^4)$$

putting in ③ we get,

$$\underbrace{\begin{bmatrix} x_1^1 & y_1^1 & 1 & 0 & 0 & 0 & -x_2^1 x_1^1 & -x_2^1 y_1^1 & -x_2^1 \\ 0 & 0 & 0 & x_1^1 & y_1^1 & 1 & -y_2^1 x_1^1 & -y_2^1 y_1^1 & -y_2^1 \\ x_1^2 & y_1^2 & 1 & 0 & 0 & 0 & -x_2^2 x_1^2 & -x_2^2 y_1^2 & -x_2^2 \\ 0 & 0 & 0 & x_1^2 & y_1^2 & 1 & -y_2^2 x_1^2 & -y_2^2 y_1^2 & -y_2^2 \\ x_1^3 & y_1^3 & 1 & 0 & 0 & 0 & -x_2^3 x_1^3 & -x_2^3 y_1^3 & -x_2^3 \\ 0 & 0 & 0 & x_1^3 & y_1^3 & 1 & -y_2^3 x_1^3 & -y_2^3 y_1^3 & -y_2^3 \\ x_1^4 & y_1^4 & 1 & 0 & 0 & 0 & -x_2^4 x_1^4 & -x_2^4 y_1^4 & -x_2^4 \\ 0 & 0 & 0 & x_1^4 & y_1^4 & 1 & -y_2^4 x_1^4 & -y_2^4 y_1^4 & -y_2^4 \end{bmatrix}}_{\text{knowns}} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ 1 \end{bmatrix}}_{\text{unknowns}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Homography in Image Stitching / Panoramas :-



$\boxed{\text{Mathematics}}$:-

Least Squares Problem :-

$$Ah = 0$$

known unknown

Linear Model Approximation :-

Linear Model

$$y = A x$$

y: $m \times 1$

A: $m \times n$ matrix

x: $n \times 1$ (unknown)

m = # of equations

n = # of unknowns

If $m = n$ and A is invertible.

$$\boxed{x = A^{-1} y}$$

If $m > n$

# of equations > # of unknowns

$\Rightarrow$ overdetermined system.

Noise in the system.

$\Rightarrow$ No exact solutions.

Maximum - Likelihood Solution :-

$$y = Ax + n$$

n: noise

$$\Rightarrow y \neq Ax$$

$$y - Ax = e$$

e: error vector (Approximation)

To find best x,

minimize approximation error

$$\min \|\bar{e}\|$$
$$= \min \|y - Ax\|$$
$$= \min \|y - Ax\|_2^2$$

(<u>Least Squares Problem</u>)

$$\min \|y - Ax\|^2 = (y - Ax)^T . (y - Ax)$$
$$= (y^T - x^T A^T) . (y - Ax)$$
$$= y^T y - \underline{x^T A^T y} - \underline{y^T A x} + x^T A^T A x$$
$$f(x) = y^T y - 2x^T A y + x^T A^T A x$$

Taking gradient,

$$\nabla_x f(x) = 0 - 2A^T y + 2A^T A x$$
$$A^T A x = A^T y$$
$$\boxed{x = (A^T A)^{-1} A^T y}$$

Least squares solution assuming $A^T A$ is invertible.

Quiz Question :-

How many degrees of freedom is Scaling, Rotation, Translation and Affine Transformation?

(a) [2, 1, 2, 4]

(b) [2, 4, 4, 5]

(c) [1, 4, 4, 6]

(d) [2, 1, 2, 6]