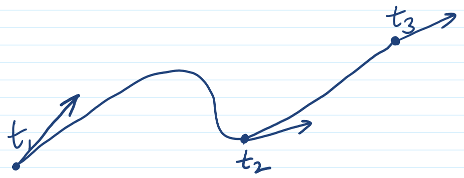
Extending this idea to set of n points.



Joining of segments
2 SEGMENTS: P_1 P_2 P_3 (Points)
P_1' P_2' P_3' (Tangents)

where P_2 and P_2' are the intermediate point and its tangent vector which is determined through some continuity contraint.

* Piecewise spline of degree k has continuity of order (k-1) at the internal joints.

Thus cubic splines have second order continuity i.e. $P_2''(t)$ is continuous over the joint.

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3$$
$$= \sum_{i=1}^{4} B_i t^{i-1} \qquad t_1 \le t \le t_2$$

$$P'(t) = B_2 + 2B_3 t + 3B_4 t^2$$
$$= \sum_{i=1}^{4} (i-1) B_i t^{i-2}$$

$$P''(t) = 2B_3 + 6B_4 t$$
$$= \sum_{i=1}^{4} (i-1)(i-2) B_i t^{i-3} \qquad t_1 \le t \le t_2$$

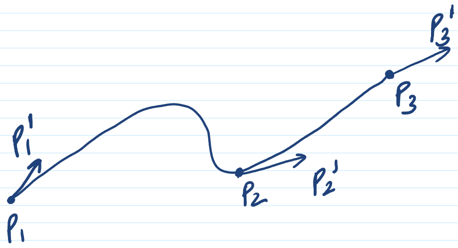
First Segment ($t = t_2$)

$P'' = 6B_4 t_2 + 2B_3$

Second Segment ($t = 0$)

$P'' = 2B_3$

So, $(6B_4 t_2 + 2B_3)_{seg\,1} = (2B_3)_{seg\,2}$ —— 1



We know,

$$\left.\begin{aligned}
B_1 &= P_1 \\
B_2 &= P_1' \\
B_3 &= \frac{3(P_2-P_1)}{t_2^2} - \frac{2(P_1')}{t_2} - \frac{P_2'}{t_2} \\
B_4 &= \frac{2(P_1-P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}
\end{aligned}\right\} \text{(A)}$$

$$(6\,B_4 t_2 + 2B_3)_{seg\,1} = (2\,B_3)_{seg\,2} \quad \text{—(1)}$$

Substituting B_3 and B_4 from (A) in (1)

$$3\left[\frac{2(P_1-P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right]t_2 + \left[\frac{3(P_2-P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right]$$

$$= \left[\frac{3(P_3-P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3}\right]$$

$$\frac{6(P_1-P_2)}{t_2^2} + \frac{3P_1'}{t_2} + \frac{3P_2'}{t_2} + \frac{3(P_2-P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}$$

$$= \frac{3(P_3-P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3}$$

$$\frac{3P_1'}{\underline{t_2}} + \frac{3P_2'}{\underline{t_2}} - \frac{2P_1'}{\underline{t_2}} - \frac{P_2'}{\underline{t_2}} + \frac{2P_2'}{t_3} + \frac{P_3'}{t_3}$$

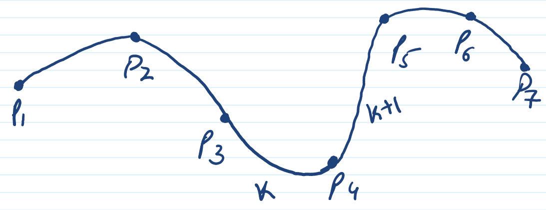
$$= \frac{3(P_3-P_2)}{t_3^2} - \frac{6(P_1-P_2)}{t_2^2} - \frac{3(P_2-P_1)}{t_2^2}$$

$$t_2 t_3 \times \left[\frac{P_1'}{t_2} + \frac{2P_2'}{t_2} + \frac{2P_2'}{t_3} + \frac{P_3'}{t_3}\right] = \frac{3(P_3-P_2)}{t_3^2} + \frac{3(P_2-P_1)}{t_2^2}$$

$$t_3 P_1' + 2(t_3+t_2)P_2' + t_2 P_3' = \frac{3t_2(P_3-P_2)}{t_3} + \frac{3t_3(P_2-P_1)}{t_2}$$

$$t_3 P_1' + 2(t_3+t_2)P_2' + t_2 P_3' = \frac{3}{t_2 t_3}\left(t_2^2(P_3-P_2) + t_3^2(P_2-P_1)\right)$$

$$\begin{bmatrix} t_3 & 2(t_3+t_2) & t_2 \end{bmatrix}\begin{bmatrix} P_1' \\ P_2' \\ P_3' \end{bmatrix} = \frac{3}{t_2 t_3}\left(t_2^2(P_3-P_2) + t_3^2(P_2-P_1)\right)$$



In general, for the k^{th} and $(k+1)^{th}$ segment $(1 \le k \le n-2)$

$$\begin{bmatrix} t_{k+2} & 2(t_{k+1}+t_{k+2}) & t_{k+1} \end{bmatrix} \begin{bmatrix} P_k' \\ P_{k+1}' \\ P_{k+2}' \end{bmatrix}$$

$$= \frac{3}{t_{k+1}\,t_{k+2}} \left(t_{k+1}^2 \left(P_{k+2} - P_{k+1}\right) + t_{k+2}^2 \left(P_{k+1} - P_k\right) \right)$$

Set of $n-2$ equations form a linear system for the tangent vectors P_k'

$$\left. \begin{array}{l} \begin{bmatrix} t_3 & 2(t_2+t_3) & t_2 & 0 & \cdots \\ 0 & t_4 & 2(t_3+t_4) & t_3 & \vdots \\ \cdots & \cdots & t_m & 2(t_m+t_{m-1}) & t_{m-1} \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_n' \end{bmatrix} \\ = \begin{bmatrix} \frac{3}{t_2 t_3}\left(t_2^2(P_3-P_2) + t_3^2(P_2-P_1)\right) \\ \frac{3}{t_3 t_4}\left(t_3^2(P_4-P_3) + t_4^2(P_3-P_2)\right) \\ \vdots \\ \frac{3}{t_{m-1} t_m}\left(t_{m-1}^2(P_m-P_{m-1}) + t_m^2(P_{m-1}-P_{m-2})\right) \end{bmatrix} \end{array} \right\} \text{B}$$

This system of equations can be used to solve for the tangent vectors $P_1', P_2' --- P_n'$

Solving for B_1 B_2 B_3 and B_4

$$B_{1k} = P_{k1}$$

$$B_{2k} = P'_{k1}$$

$$B_{3k} = \frac{3(P_{k+1} - P_k)}{t_{k+1}^2} - \frac{2P'_k}{t_{k+1}} - \frac{P'_{k+1}}{t_{k+1}}$$

$$B_{4k} = \frac{2(P_k - P_{k+1})}{t_{k+1}^3} + \frac{P'_k}{t_{k+1}^2} + \frac{P'_{k+1}}{t_{k+1}^2}$$

Rearranging,

$$\begin{bmatrix} B_{1k} \\ B_{2k} \\ B_{3k} \\ B_{4k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/t_{k+1}^3 & 1/t_{k+1}^2 \end{bmatrix} \begin{bmatrix} P_k \\ P'_k \\ P_{k+1} \\ P'_{k+1} \end{bmatrix}$$

$$P_k(t) = \sum_{i=1}^{4} B_{ik} t^{k-1} \qquad 0 \le t \le t_{k+1}$$

$$1 \le k \le n-1$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} B_{1k} & B_{2k} & B_{3k} & B_{4k} \end{bmatrix}^T$$

$$P_k(t) = \underset{1\times4}{\begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}} \underset{4\times4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/t_{k+1}^3 & 1/t_{k+1}^2 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} P_k \\ P'_k \\ P_{k+1} \\ P'_{k+1} \end{bmatrix}}$$

$$P_k(t) = \underset{1\times4}{\left[\left(1 - \frac{3t^2}{t_{k+1}^2} + \frac{2t^3}{t_{k+1}^3}\right) \quad \left(t - \frac{2t^2}{t_{k+1}} + \frac{t^3}{t_{k+1}^2}\right) \quad \left(\frac{3t^2}{t_{k+1}^2} - \frac{2t^3}{t_{k+1}^3}\right) \quad \left(\frac{-t^2}{t_{k+1}} + \frac{t^3}{t_{k+1}^2}\right) \right]} \underset{4\times1}{\begin{bmatrix} P_k \\ P'_k \\ P_{k+1} \\ P'_{k+1} \end{bmatrix}}$$

$$\begin{bmatrix} \\ P'_{k+1} \end{bmatrix}_{4\times 1}$$

Substituting $u = t/t_{k+1}$ rearranging

$$P_k(u) = \begin{bmatrix} F_1(u) & F_2(u) & F_3(u) & F_4(u) \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P'_k \\ P'_{k+1} \end{bmatrix}$$

$$0 \le u \le 1$$
$$1 \le k \le n-1$$

$$F_1(u) = 2u^3 - 3u^2 + 1$$
$$F_2(u) = u(u^2 - 2u + 1)\, t_{k+1}$$
$$F_3(u) = -2u^3 + 3u^2$$
$$F_4(u) = u(u^2 - u)\, t_{k+1}$$

where F_1, F_2, F_3, F_3 are called the Blending Functions.