

curves

Higher degree approximations :-

① Explicit  
 $y = f(x)$

② Implicit  
 $f(x, y) = 0$

Defines curves implicitly as solution of equation system.

line :  $ax + by + c = 0$

circle :  $x^2 + y^2 - R^2 = 0$

③ Parametric

$$x = x(t), y = y(t)$$

Position on the curve is defined through a parameter.

Parametric curves form a rich variety of free form smooth curves.

"Splines"

Cubic Splines :-

$$P(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$
$$= \sum_{i=1}^4 b_i t^{i-1} \quad t_1 \leq t \leq t_2$$

is defined as a cubic polynomial of the parameter  $t$ .

$$\Rightarrow x(t) = \sum_{i=1}^4 b_{x_i} t^{i-1}$$

$$y(t) = \sum_{i=1}^4 b_{y_i} t^{i-1}$$

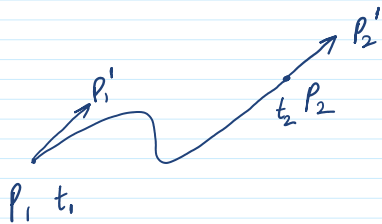
Vectors

$$P'(t) = b_1 + 2b_2 t + 3b_3 t^2$$

Tangent vectors at some location.

Example:-

Given,



Two end points  $P_1$  and  $P_2$  of the cubic curve.  
(position vectors)

$P_1'$  and  $P_2'$  are the tangent vectors at point  $P_1$  and  $P_2$ .

These points are defined for the parameter values  $t_1$  and  $t_2$ .

Let  $t_1 = 0$

$$\begin{aligned} P(0) &= P_1 & P(t_2) &= P_2 \\ P'(0) &= P_1' & P'(t_2) &= P_2' \end{aligned}$$

We know,

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3$$

$$P'(t) = B_2 + 2B_3 t + 3B_4 t^2$$

To obtain the generalized curve,

$$P(0) = B_1 = P_1$$

$$P(t_2) = B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2$$

$$P'(0) = B_2 = P_1'$$

$$P'(t_2) = B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P_2'$$

We get,

$$B_1 = P_1 \quad \text{--- (1)}$$

$$B_2 = P_1' \quad \text{--- (2)}$$

$$B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2 \quad \text{--- (3)}$$

$$B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P_2' \quad \text{--- (4)}$$

Using ① & ② in ③ & ④

$$P_1 + P_1' t_2 + B_3 t_2^2 + B_4 t_2^3 = P_2$$

$$P_1' + 2B_3 t_2 + 3B_4 t_2^2 = P_2'$$

solving for  $B_3$  and  $B_4$ ,

$$\text{Assuming } \left. \begin{array}{l} P_1 \rightarrow a \quad P_1' \rightarrow b \quad t_2 \rightarrow c \\ P_2 \rightarrow d \quad P_2' \rightarrow e \\ B_3 \rightarrow x \quad B_4 \rightarrow y \end{array} \right\} \text{ for simplicity}$$

$$a + bc + \cancel{x c^2} + y c^3 = d$$

$$\frac{c}{2} \times [ b + 2\cancel{x c} + 3y c^2 = e ]$$

$$\frac{bc}{2} + \cancel{\frac{x c^2}{2}} + \frac{3y c^3}{2} = \frac{ec}{2}$$

$$a + bc - \frac{bc}{2} + y c^3 - \frac{3y c^3}{2} = d - \frac{ec}{2}$$

$$a + \frac{bc}{2} - \frac{y c^3}{2} = d - \frac{ec}{2}$$

$$-\frac{y c^3}{2} = d - \frac{ec}{2} - a - \frac{bc}{2}$$

$$y = -\frac{2}{c^3} \left[ d - \frac{ec}{2} - a - \frac{bc}{2} \right]$$

$$y = -\frac{2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2}$$

$$b + 2xc + 3 \left[ -\frac{2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2} \right] c^2 = e$$

$$b + 2xc - \frac{6d}{c} + \frac{3ec}{c^2} + \frac{6ac}{c^3} + \frac{3b}{c^2} = e$$

$$b + 2uc - \frac{6d}{c^3} + \frac{3e}{c^2} + \frac{6a}{c^3} + \frac{3b}{c^2} = e$$

$$2uc - \frac{6d}{c} + 2e + \frac{6a}{c} + 4b = 0$$

$$2uc = \frac{6d - 6a - 2e - 4b}{c}$$

$$uc = \frac{3d - 3a - e - 2b}{c}$$

$$u = \frac{3d - 3a - e - 2b}{c^2}$$

$$P_1 \rightarrow a \quad P_1' \rightarrow b \quad t_2 \rightarrow c$$

$$P_2 \rightarrow d \quad P_2' \rightarrow e$$

$$B_3 \rightarrow u \quad B_4 \rightarrow y$$

$$B_3 = \frac{3P_2}{t_2^2} - \frac{3P_1}{t_2^2} - \frac{P_2'}{t_2} - \frac{2P_1'}{t_2}$$

$$B_3 = \frac{3P_2 - 3P_1}{t_2^2} - \frac{2P_1' - P_2'}{t_2}$$

$$y = -\frac{2d}{c^3} + \frac{e}{c^2} + \frac{2a}{c^3} + \frac{b}{c^2}$$

$$B_4 = \frac{-2P_2}{t_2^3} + \frac{P_2'}{t_2^2} + \frac{2P_1}{t_2^3} + \frac{P_1'}{t_2^2}$$

$$= \frac{2P_1}{t_2^3} - \frac{2P_2}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}$$

$$B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}$$

Solving for  $B_1, B_2, B_3$  and  $B_4$

$$B_1 = P_1$$

$$B_2 = P_1'$$

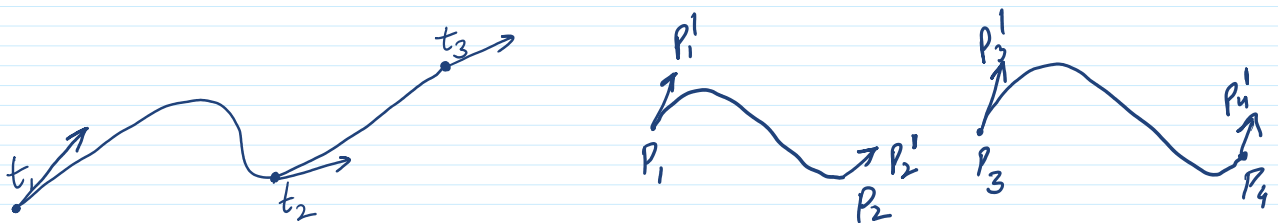
$$B_3 = 3(P_2 - P_1) \quad 2(P_1') \quad 0' \quad \text{P (A)}$$

$$\begin{aligned}
 b_2 &= P_1 \\
 b_3 &= \frac{3(P_2 - P_1)}{t_2^2} - \frac{2(P_1')}{t_2} - \frac{P_2'}{t_2} \\
 b_4 &= \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}
 \end{aligned}
 \quad \text{(A)}$$

where  $P_1$  and  $P_2$  gives the position of the endpoints.  
and  $P_1'$  and  $P_2'$  gives the direction of the tangent vectors.

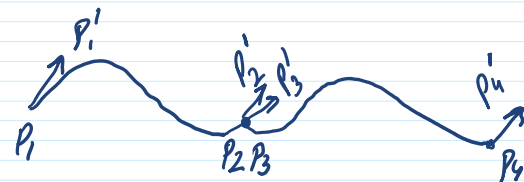
$$\begin{aligned}
 P(t) &= P_1 + P_1' t + \left( \frac{3(P_2 - P_1)}{t_2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2} \right) t^2 \\
 &\quad + \left( \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2} \right) t^3
 \end{aligned}$$

Extending this idea to set of  $n$  points.



Joining of segments

2 SEGMENTS:  $P_1, P_2, P_3$  (Points)  
 $P_1', P_2', P_3'$  (Tangents)



where  $P_2$  and  $P_2'$  are the intermediate point and its tangent vector which is determined through some continuity constraint.

\* Piecewise spline of degree  $k$  has continuity of order  $(k-1)$  at the internal joints.

Thus cubic splines have second order continuity  
i.e.  $P_2''(t)$  is continuous over the joint.

$$P''(t) = \sum_{i=1}^4 (i-1)(i-2) B_i t^{i-3} \quad t_1 \leq t \leq t_2$$

at  $t = t_2$

First Segment

$$P'' = 6B_4 t_2 + 2B_3$$

Second Segment

$$P'' = 2B_3$$

$$\text{So, } (6B_4 t_2 + 2B_3)_{\text{seg 1}} = (2B_3)_{\text{seg 2}}$$

Substitute the expressions for  $B_4$  and  $B_3$  and rearranging